

Symmetries of geodesic motion in Gödel-type spacetimes

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Abstract. In this paper, we study Noether gauge symmetries of geodesic motion for geodesic Lagrangian of four classes of metrics of Gödel-type spacetimes for which we calculated the Noether gauge symmetries for all classes I-IV, and find the first integrals of corresponding classes to derive a complete characterization of the geodesic motion. Using the obtained expressions for $\dot{t}, \dot{r}, \dot{\phi}$ and \dot{z} of each classes I-IV which depends essentially on two independent parameters m and w , we explicitly integrated the geodesic equations of motion for the corresponding Gödel-type spacetimes.

Keywords: Gödel-type spacetime, geodesic equation, Noether gauge symmetry

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1 Introduction

In 1949, Kurt Gödel [1] derived an exact cosmological solution of Einstein’s field equations, in which the rotation of a homogeneous mass distribution around every point is presented with a constant rotation rate. The Gödel’s metric is the best known example of causality violated universe model [2]. The existence of closed timelike curves (CTCs) is particular property of Gödel’s universe. The rotational symmetry of Gödel’s metric comes from the existence of CTCs corresponding to circular orbits in specific coordinates, as pointed out by Gödel [1]. Furthermore these circular orbits have discussed by Raychaudhuri and Thakurta [3]. Rebouças and Tiomno [4], and Calvão et al [5] have pointed out that the causality features of the Gödel-type spacetimes depend on two independent parameters: m and w . For $0 \leq m^2 < 4w^2$, they have shown that there exists only one non-causal region. For $m^2 \geq 4w^2$, there is no CTCs, in which the limiting case $m^2 = 4w^2$ yields a completely causal and spacetime homogeneous Gödel model; for $m^2 < 0$, there exists an infinite number of alternating causal and noncausal regions. The Gödel metrics are mainly interesting for their high degree of symmetry [4, 6–8]. All classes of Gödel-type spacetimes admit at least a G_5 group of motions. In a special case $m^2 = 4w^2$, it has been shown that the group of motions is G_7 , a maximal symmetry group of Gödel-type spacetimes [6].

Applying the method of effective potential to the Schwarzschild and Kerr metrics the qualitative features of their geodesics have been explored [9, 10]. The geodesic equations of motion for the general cylindrically symmetric stationary spacetimes together with their Dirac’s constraint analysis and symplectic structure have been obtained, and integrated in Ref. [11]. The geodesic equations of motion in Gödel-type spacetimes have been analyzed by several authors. The geodesic equations for Gödel’s metric was firstly solved by Kundt [12], who used the Killing vectors and corresponding constants of motion. Chandrasekhar and Wright [13] have presented an independent derivation of the solution for the geodesic equations of Gödel’s metric. Novello *et.al.* [14] have provided a detailed discussion on geodesic motion in the original Gödel’s universe using the method of effective potential as well as the analytical solution. Rebouças and Tiomno [15] have integrated the geodesic equations for the special case $m^2 = 4w^2$ with seven isometries, where the spacetime is conformally flat and Petrov type

O. Paiva *et. al.* [16] have examined the geodesics of the Som-Raychaudhuri spacetime [17]. Calvão *et. al.* [18] followed Novello *et. al.* [14] and give a complete discussion of timelike geodesics and also treat null geodesics for Gödel-type spacetimes. Grave *et. al.* [19] derived the analytical solution of the geodesic equations of Gödel's universe for both particles and light in a special set of coordinates. They have generalized the work of Kajari *et. al.* [20] on the solution of lightlike geodesic equations. Recently, Dautcourt [21] considered only the lightlike case, and studied the lightcone of the Gödel-type metrics. Some spacetime symmetry properties of the original Gödel metric and Gödel-type spacetimes (see Refs.[22]-[26]) will be discussed in the following section.

In order to solve the geodesic equations of motion, the central idea is to find simple expressions for constants of motion, i.e. conservation laws. To derive the equations of geodesic motion, one can use the Lagrangian formalism. The Noether symmetries are interesting symmetries associated with differential equations possessing a Lagrangian, and they describe physical features of differential equations in terms of conservation laws admitted by them. The strict Noether symmetry approach which does not include a gauge term (see Refs.[27]-[35]) is a kind of symmetry in which the Lie derivative of Lagrangian arising from the metric of interest dragging along a vector field \mathbf{Y} vanishes, i.e. $\mathcal{L}_{\mathbf{Y}}L = 0$. The Noether gauge symmetry (NGS) approach [36]-[44], as a generalization of the former strict Noether symmetry approach, will be discussed in section 4. The connection between the KVs and NGSs of spacetimes is examined by several works. For the spaces of different curvatures such as maximally symmetric spacetimes and Bertotti-Robinson like spacetime, the existence of new conserved quantities has been discussed and conjectured [36–38]. The NGSs of FRW spacetimes have been studied by Tsamparlis and Paliathanasis [39]. They have also examined the NGSs of class A Bianchi type homogeneous spacetimes with a scalar field that is minimally coupled to the gravity [40]. Recently, Ali and Feroze [41] have provided a classification of plane symmetric static spacetimes according to their geodesic Lagrangian considering NGS approach.

Physically, the presence of conserved quantities which is directly related with Noether symmetries gives selection rule to recover classical behaviour in cosmic evolution, and they reduce the number of dynamical variables of the system due to the cyclic variables appeared. Basic geometrical symmetries, namely Lie point and Noether ones connected to the spacetimes like FRW and Bianchi type universe models have been discussed in context of $f(R)$, $f(T)$ and scalar-tensor gravity theories (see [27]-[35] and [45]). Furthermore, the existence of Noether symmetries yields a classification of point singularities, where the symmetry is broken, for cosmologies coming from the extended theories of gravity (see the review of [46]). In the cosmological contexts, the Noether symmetry technique can play a crucial role. For example, in any gravity theory including a scalar field, the Noether symmetry gives us a specific form of coupling function and the potential [42–44]. An easy way for explaining the accelerated expansion of the universe is usually to consider the dark energy with negative pressure, and the simplest dark energy candidate takes place of the cosmological constant. Recently, it is proposed to use the Noether symmetry approach to probe the nature of dark energy [47].

This study is designed as follows. In the following section, we give a short review about Gödel-type spacetimes and their properties. In section 3, we present the equations of geodesic motion for Gödel-type spacetimes and their geodesic Lagrangian and Hamiltonian structure. In section 4, we explicitly discuss the Noether gauge symmetry approach of geodesic Lagrangian for Gödel-type spacetimes while in section 5, we give solution of Noether gauge symmetry equations in detail. Finally, our conclusion with a brief summary and discussions of finding is presented in Section 6.

2 Gödel-type Spacetimes

In cylindrical coordinates, $x^a = (t, r, \phi, z)$, $a = 0, 1, 2, 3$, the line element for the Gödel-type spacetimes can be written [2, 3]

$$ds^2 = [dt + H(r)d\phi]^2 - dr^2 - D^2(r)d\phi^2 - dz^2. \quad (2.1)$$

The necessary and sufficient conditions for a Gödel-type manifold to be spacetime homogeneous (STH, hereafter) are found as [4–7]

$$\frac{D''}{D} = \text{const} \equiv m^2, \quad (2.2)$$

$$\frac{H'}{D} = \text{const} \equiv -2\omega \quad (2.3)$$

where prime denotes derivative with respect to the radial coordinate r . All STH Riemannian manifolds endowed with a Gödel-type spacetime (2.1) are obtained as follows:

Class I : $m^2 > 0, \omega \neq 0$. For this case, the general solution of Eqs. (2.2) and (2.3) can be written by

$$H(r) = \frac{2\omega}{m^2} [1 - \cosh(mr)] \quad \text{and} \quad D(r) = \frac{1}{m} \sinh(mr). \quad (2.4)$$

Class II : $m^2 = 0, \omega \neq 0$. The general solution of Eqs. (2.2) and (2.3) is

$$H(r) = -\omega r^2 \quad \text{and} \quad D(r) = r, \quad (2.5)$$

where only the essential parameter ω appears.

Class III : $m^2 \equiv -\mu^2 < 0, \omega \neq 0$. Similarly, the integration of the conditions for homogeneity Eqs. (2.2) and (2.3) leads to

$$H(r) = \frac{2\omega}{\mu^2} [\cos(\mu r) - 1] \quad \text{and} \quad D(r) = \frac{1}{\mu} \sin(\mu r). \quad (2.6)$$

Class IV : $m^2 \neq 0, \omega = 0$. We refer to the manifolds of this class as degenerated Gödel-type manifolds, since the cross term in the line element, related to the rotation ω in the Gödel model, vanishes. By a trivial coordinate transformation, one can make $H = 0$ with $D(r)$ given, respectively, by Eqs. (2.4) or (2.6) depending on whether $m^2 > 0$ or $m^2 \equiv -\mu^2 < 0$. Throughout this paper we have used the following property

$$D^2 \left(\frac{D'}{D} \right)' = -1, \quad (2.7)$$

which is valid for STH Gödel-type spacetimes only. The form of the fully-covariant Riemann curvature tensor, Weyl tensor and Ricci tensor for the Gödel-type spacetime, with non-vanishing components are as follows

$$R_{0101} = w^2, \quad R_{0202} = w^2 D^2, \quad R_{0112} = -w^2 H^2, \quad R_{1212} = w^2 H^2 + \left(\frac{3w^2}{4} - m^2 \right) D^2, \quad (2.8)$$

$$C_{0101} = \frac{1}{6}(m^2 - 4w^2), \quad C_{0202} = D^2 C_{0101}, \quad C_{0303} = -2C_{0101},$$

$$C_{0112} = -H C_{0101}, \quad C_{1212} = (H^2 + 2D^2) C_{0101}, \quad C_{1313} = -C_{0101}, \quad (2.9)$$

$$C_{2303} = -2H C_{0101}, \quad C_{2323} = (2H^2 + D^2) C_{0101}$$

$$R_{00} = 2w^2, \quad R_{11} = m^2 - 2w^2, \quad R_{02} = -2w^2 H, \quad R_{22} = m^2 D^2 - 2w^2 (H^2 + D^2), \quad (2.10)$$

Thus the scalar curvature R becomes $R = 2(w^2 - m^2)$. The results in Refs. [3, 4, 6, 7] for Gödel-type manifolds can be collected together as follows :

Theorem 1 : The necessary and sufficient conditions for a four-dimensional Riemannian Gödel-type manifold to be locally homogeneous are those given by Eqs. (2.2) and (2.3).

Theorem 2 : The four-dimensional homogeneous Riemannian Gödel-type manifolds are locally characterized by two independent parameters m^2 and ω : the pair of (m^2, ω) identically specify locally equivalent manifolds.

Now let us recall the spacetime symmetries. If \mathbf{X} be any global smooth vector field and g_{ab} the metric tensor field of any type on manifold M , then the natural concept of a symmetry is geometrically given as a mapping, and reduced to a differential relation between \mathbf{X} and g_{ab} as [22]

$$\mathcal{L}_{\mathbf{X}}g_{ab} = 2\psi g_{ab} \quad (2.11)$$

where $\mathcal{L}_{\mathbf{X}}$ is the Lie derivative operator along the vector field \mathbf{X} , and $\psi = \psi(x^a)$ is a conformal factor. The group of conformal motions generated by a *conformal Killing vector* (CKV) field \mathbf{X} is defined by Eq. (2.11). For $\psi_{;ab} \neq 0$, the CKV field \mathbf{X} is said to be *proper*, otherwise it is a special conformal Killing vector (SCKV) field ($\psi_{;ab} = 0$). The vector field \mathbf{X} is a homothetic vector (HV) for $\psi_{,a} = 0$, and it is an isometry or a Killing vector (KV) field for $\psi = 0$. The set of all CKV (respectively SCKV, HKV and KV) form a finite-dimensional Lie algebra denoted by \mathcal{C} (respectively \mathcal{S}, \mathcal{H} and \mathcal{G}).

The KV fields as well as corresponding Lie algebra of the classes I-IV of STH Gödel-type spacetimes (2.1) have been determined by Rebouças *et al.* [8], and the results are stated in the following theorem:

Theorem 3 : The four-dimensional homogeneous Riemannian Gödel-type manifolds admit a group of isometry G_r with

- (i) $r = 5$ in classes I (where $m^2 \neq 4\omega^2$) , II and III;
- (ii) $r = 6$ in class IV;
- (iii) $r = 7$ in the special case of class I, where $m^2 = 4\omega^2$.

It has been pointed out that the original Gödel metric does not admit HVs [23], of which the result is subsequently extended to the STH Gödel-type spacetimes [24]. The proper CKVs and complete conformal algebra of a Gödel-type spacetime have been computed in Ref. [25]. The Ricci collineations (RCs) and contracted RCs of STH Gödel-type spacetimes are studied by Melfo *et al.* [24]. The matter collineations (MCs) of that spacetimes are obtained by Camci and Sharif [26]. In this work, we aim to give a complete classification for STH Gödel-type spacetimes according to the Noether gauge symmetries of their geodesic Lagrangian.

3 The Equations of Geodesic Motion

For any spacetime metric, the freely moving massive particles or the propagation of light rays is described by geodesic equations of motion

$$\ddot{x}^a + \Gamma_{bc}^a \dot{x}^b \dot{x}^c = F^a \quad (3.1)$$

with the following constraint to be fulfilled

$$g_{ab} \dot{x}^a \dot{x}^b = \epsilon, \quad (3.2)$$

where the functions $\Gamma_{bc}^a(x^e)$ are the connection coefficients of the metric, a dot over a symbol represents a derivative with respect to proper time τ (for massive particle motion) or with

regard to an affine parameter λ (for lightlike geodesics) along the solution curve, and $F^a(x^e)$ is the conservative force field. Here, we can write the force field as $F^a(x^e) = g^{ab}V_{,b}$, where $V(x^e)$ is the potential function. We have $\epsilon = -1, 0, +1$ for spacelike, lightlike (or null) and timelike geodesics, respectively.

Using the Gödel-type spacetime (2.1), a point-like Lagrangian density takes such a form

$$L = \frac{1}{2} \left[\dot{t}^2 - \dot{r}^2 - \dot{z}^2 + (H^2(r) - D^2(r))\dot{\phi}^2 \right] + H(r)\dot{t}\dot{\phi} - V(t, r, \phi, z). \quad (3.3)$$

One may obtain the Euler-Lagrange equations of motion by varying of the Lagrangian (3.3) with respect to the coordinates t, r, ϕ and z as given by

$$\ddot{t} + H\ddot{\phi} + H'\dot{r}\dot{\phi} + V_{,t} = 0, \quad (3.4)$$

$$\ddot{r} + (HH' - DD')\dot{\phi}^2 + H'\dot{r}\dot{\phi} - V_{,r} = 0, \quad (3.5)$$

$$H\ddot{t} + (H^2 - D^2)\ddot{\phi} + 2(HH' - DD')\dot{r}\dot{\phi} + H'\dot{r}\dot{t} - V_{,\phi} = 0, \quad (3.6)$$

$$\ddot{z} - V_{,z} = 0. \quad (3.7)$$

The *energy functional* or *Hamiltonian of the dynamical system*, E_L , associated with the Lagrangian (3.3) is found as

$$\begin{aligned} E_L &= \dot{t}\frac{\partial L}{\partial \dot{t}} + \dot{r}\frac{\partial L}{\partial \dot{r}} + \dot{\phi}\frac{\partial L}{\partial \dot{\phi}} + \dot{z}\frac{\partial L}{\partial \dot{z}} - L \\ &= \frac{1}{2} \left[\dot{t}^2 - \dot{r}^2 - \dot{z}^2 + (H^2 - D^2)\dot{\phi}^2 \right] + H\dot{t}\dot{\phi} + V(t, r, \phi, z). \end{aligned} \quad (3.8)$$

Introducing the momenta $p_a = g_{ab}\dot{x}^b = \frac{\partial L}{\partial \dot{x}^a}$ we have $p_t = \dot{t} + H\dot{\phi}$, $p_r = -\dot{r}$, $p_\phi = H\dot{t} + (H^2 - D^2)\dot{\phi}$, $p_z = -\dot{z}$. Then E_L becomes

$$\begin{aligned} E_L &= \dot{x}^a p_a - L \\ &= \frac{1}{2} \left[p_t^2 - p_r^2 - p_z^2 - \frac{1}{D^2} (p_\phi - Hp_t)^2 \right] + V(t, r, \phi, z). \end{aligned} \quad (3.9)$$

4 The Noether Symmetry Approach

The *Noether gauge symmetry* (NGS) is defined as follows. Let us consider a vector field

$$\mathbf{Y} = \xi \frac{\partial}{\partial \tau} + \eta^0 \frac{\partial}{\partial t} + \eta^1 \frac{\partial}{\partial r} + \eta^2 \frac{\partial}{\partial \phi} + \eta^3 \frac{\partial}{\partial z} \quad (4.1)$$

where $\xi, \eta^1, \eta^2, \eta^3$ and η^4 are depend on τ, t, r, ϕ and z . Here, τ is an independent variable, $t(\tau), r(\tau), \phi(\tau)$ and $z(\tau)$ are dependent variables, i.e. the configuration space of the Lagrangian (3.3) is $Q = (t, r, \phi, z)$, whose tangent space is $TQ = (t, r, \phi, z, \dot{t}, \dot{r}, \dot{\phi}, \dot{z})$. The first extension of the above vector field is given by

$$\mathbf{Y}^{[1]} = \mathbf{Y} + \eta_\tau^0 \frac{\partial}{\partial \dot{t}} + \eta_\tau^1 \frac{\partial}{\partial \dot{r}} + \eta_\tau^2 \frac{\partial}{\partial \dot{\phi}} + \eta_\tau^3 \frac{\partial}{\partial \dot{z}}, \quad (4.2)$$

in which

$$\eta_\tau^0 = D_\tau \eta^0 - \dot{t} D_\tau \xi, \quad \eta_\tau^1 = D_\tau \eta^1 - \dot{r} D_\tau \xi, \quad \eta_\tau^2 = D_\tau \eta^2 - \dot{\phi} D_\tau \xi, \quad \eta_\tau^3 = D_\tau \eta^3 - \dot{z} D_\tau \xi, \quad (4.3)$$

where D_τ is the operator of total differentiation with respect to τ

$$D_\tau = \frac{\partial}{\partial \tau} + \dot{t} \frac{\partial}{\partial t} + \dot{r} \frac{\partial}{\partial r} + \dot{\phi} \frac{\partial}{\partial \phi} + \dot{z} \frac{\partial}{\partial z}. \quad (4.4)$$

The vector field \mathbf{Y} is a NGS of a Lagrangian $L(\tau, t, r, \phi, z, \dot{t}, \dot{r}, \dot{\phi}, \dot{z})$ if there exists a gauge function $f(\tau, t, r, \phi, z)$ such that

$$\mathbf{Y}^{[1]}L + L(D_\tau \xi) = D_\tau f \quad (4.5)$$

which takes the alternative form [39, 40]

$$\xi_{,a} = 0, \quad g_{ab}\eta_{,\tau}^b = f_{,a} \quad \mathcal{L}_\eta g_{ab} = \xi_{,\tau} g_{ab}, \quad \mathcal{L}_\eta V = -\xi_{,\tau} V - f_{,\tau} \quad (4.6)$$

where \mathcal{L}_η is the Lie derivative operator along $\eta = \eta^0 \partial_t + \eta^1 \partial_r + \eta^2 \partial_\phi + \eta^3 \partial_z$. It is noted here that the set of all NGSs form a finite dimensional Lie algebra denoted by \mathcal{N} .

The significance of NGS is clearly comes from the fact that if \mathbf{Y} is the Noether gauge symmetry corresponding to the Lagrangian $L(\tau, x^a, \dot{x}^a)$, then

$$I = \xi L + (\eta^a - \xi \dot{x}^a) \frac{\partial L}{\partial \dot{x}^a} - f \quad (4.7)$$

is a *first integral* or a *conserved quantity* associated with NGS vector field \mathbf{Y} [48]. Then it follows from this relation for the geodesic Lagrangian (3.3) that

$$I = -\xi E_L + (\eta^0 + H\eta^2)\dot{t} - \eta^1 \dot{r} + [H\eta^0 + (H^2 - D^2)\eta^2] \dot{\phi} - \eta^3 \dot{z} - f, \quad (4.8)$$

where E_L is given in (3.8). Now we seek the condition in order that the Lagrangian (3.3) would admit NGS.

Recently the Noether gauge symmetries of geodesic Lagrangian for some spacetimes have been calculated, and classified according to their symmetry generators [36]-[40]. Here, we investigate the NGSs of the Gödel-type spacetimes. For the Gödel-type spacetimes (2.1), the Noether gauge symmetry equations (4.5) or (4.6) yield 19 partial differential equations:

$$\xi_{,t} = 0, \quad \xi_{,r} = 0, \quad \xi_{,\phi} = 0, \quad \xi_{,z} = 0, \quad (4.9)$$

$$2T_{,t} - \xi_{,\tau} = 0, \quad T_{,z} - Z_{,t} = 0, \quad 2R_{,r} - \xi_{,\tau} = 0, \quad (4.10)$$

$$Z_{,r} + R_{,z} = 0, \quad 2Z_{,z} - \xi_{,\tau} = 0, \quad (4.11)$$

$$T_{,r} - R_{,t} - \frac{H'}{D}P = 0, \quad (4.12)$$

$$Z_{,\phi} + DP_{,z} - H Z_{,t} = 0, \quad (4.13)$$

$$T_{,\phi} + H'R - DP_{,t} - \frac{H}{2}\xi_{,\tau} = 0, \quad (4.14)$$

$$R_{,\phi} - HR_{,t} - D'P + DP_{,r} = 0, \quad (4.15)$$

$$P_{,\phi} - HP_{,t} + D'R - \frac{D}{2}\xi_{,\tau} = 0, \quad (4.16)$$

$$T V_{,t} + R V_{,r} + \frac{1}{D}(V_{,\phi} - H V_{,t})P + Z V_{,z} + \xi_{,\tau} V + f_{,\tau} = 0, \quad (4.17)$$

$$T_{,\tau} - f_{,t} = 0, \quad R_{,\tau} + f_{,r} = 0, \quad DP_{,\tau} + f_{,\phi} - H f_{,t} = 0, \quad Z_{,\tau} + f_{,z} = 0, \quad (4.18)$$

where the subscripts with comma denotes partial derivatives, and we have used the following definition

$$\eta^0 = T - \frac{H}{D}P, \quad \eta^1 = R, \quad \eta^2 = \frac{P}{D}, \quad \eta^3 = Z. \quad (4.19)$$

The general solution to the above NGS equations is introduced in the next section for each classes I-IV for the Gödel-type spacetimes.

5 The Solution of Noether Symmetry Equations

After some algebra, we have calculated the general solution to Eqs. (4.9)-(4.18) in order to get NGSs for each of the classes I, II, III and IV when the potential function $V(x^e)$ vanishes. From the first set of equations (4.9) we have $\xi = \xi(\tau)$. From (4.10) and (4.11) one obtains

$$T = z[-rh_1(\tau, \phi) + h_2(\tau, \phi)] + f_1(\tau, r, \phi) + \frac{t}{2}\xi(\tau)_{,\tau} \quad (5.1)$$

$$R = z[th_1(\tau, \phi) + h_3(\tau, \phi)] + g_1(\tau, t, \phi) + \frac{r}{2}\xi(\tau)_{,\tau} \quad (5.2)$$

$$Z = t[-rh_1(\tau, \phi) + h_2(\tau, \phi)] - rh_3(\tau, \phi) + h_4(\tau, \phi) + \frac{z}{2}\xi(\tau)_{,\tau} \quad (5.3)$$

where $H' \neq 0$, and $h_1(\tau, \phi), h_2(\tau, \phi), h_3(\tau, \phi), h_4(\tau, \phi), f_1(\tau, r, \phi), g_1(\tau, t, \phi)$ are arbitrary integration functions of their arguments. These solutions do not depend on the metric functions $D(r)$ and $H(r)$, and so they are the general solution of Eqs. (4.10)-(4.11). The equations (5.1)-(5.3) are the general solution to the NGS equations (4.9)-(4.11) for all STH Gödel-type metrics. These general solutions depend on six arbitrary functions, which will be determined by the remaining NGS equations for each different class of STH Gödel-type Riemannian manifolds.

5.1 Classes I, II and III

Now let us try to find remaining arbitrary function $P(\tau, t, r, \phi, z)$. Using the property $H' = -2wD$ for STH Gödel-type spacetimes then the Eq.(4.12) yields

$$P = \frac{1}{2w} [g_{1,t} - f_{1,r} + 2zh_1(\tau, \phi)]. \quad (5.4)$$

Inserting (5.3) and (5.4) into (4.13) one obtains $h_1 = h_2 = 0, h_3 = h_3(\tau)$ and $h_4 = h_4(\tau)$. Thus substitution of (5.1), (5.2) and (5.4) into the Eq. (4.14) yields $h_3 = 0$, and

$$f_1 = 2w [D\ell_1(\tau, \phi) + k_1(\tau, r)] + \phi \left(wrD + \frac{H}{2} \right) \xi(\tau)_{,\tau} \quad (5.5)$$

$$g_1 = \ell_{1,\phi} + g(\tau, \phi) \sin(2wt) + h(\tau, \phi) \cos(2wt), \quad (5.6)$$

where ℓ_1, k_1, g and h are the arbitrary functions of their arguments. For vanishing potential, the Eq. (4.17) reduces to $f_{,\tau} = 0$, i.e. the gauge function is not depend on τ . Using (5.1) in the first equation of (4.18) we find that the component $\xi(\tau)$, the gauge function f , and the functions ℓ_1 and k_1 have the following forms

$$\xi = c_1 + a_1\tau + a_2\frac{\tau^2}{2}, \quad (5.7)$$

$$f = a_2\frac{t^2}{4} + t \left[2w(DL_1(\phi) + K_1(r)) + a_2\phi \left(wrD + \frac{H}{2} \right) \right] + f_2(r, \phi, z), \quad (5.8)$$

$$\ell_1 = \tau L_1(\phi) + L_2(\phi), \quad k_1 = \tau K_1(r) + K_2(r), \quad (5.9)$$

where a_1, a_2, c_1 are integration constants, and $L_1(\phi), L_2(\phi), K_1(r), K_2(r), f_2(r, \phi, z)$ are integration functions. The functions T, R, Z and P can now be simplified to give

$$T = 2w [D(\tau L_1 + L_2) + \tau K_1 + K_2] + (a_1 + a_2\tau) \left[\phi \left(wrD + \frac{H}{2} \right) + \frac{t}{2} \right], \quad (5.10)$$

$$R = \tau L_{1,\phi} + L_{2,\phi} + g(\tau, \phi) \sin(2wt) + h(\tau, \phi) \cos(2wt) + \frac{r}{2}(a_1 + a_2\tau), \quad (5.11)$$

$$Z = h_4(\tau) + \frac{z}{2}(a_1 + a_2\tau), \quad (5.12)$$

$$P = g(\tau, \phi) \cos(2wt) - h(\tau, \phi) \sin(2wt) - D'(\tau L_1 + L_2) - \tau K_{1,r} - K_{2,r} - \phi \frac{rD'}{2}(a_1 + a_2\tau). \quad (5.13)$$

If we substitute (5.11) and (5.13) into (4.15), then the resulting equations to be satisfied are given by

$$L_{1,\phi\phi} + L_1 - DK_{1,rr} + D'K_{1,r} + a_2 \frac{\phi}{2}(r - DD') = 0, \quad (5.14)$$

$$L_{2,\phi\phi} + L_2 - DK_{2,rr} + D'K_{2,r} + a_1 \frac{\phi}{2}(r - DD') = 0, \quad (5.15)$$

$$g_{,\phi} + (D' + 2wH)h = 0, \quad (5.16)$$

$$h_{,\phi} - (D' + 2wH)g = 0. \quad (5.17)$$

It can be easily seen from Eqs. (5.14) and (5.15) that for the classes I and III, $a_1 = a_2 = 0$. In class II (where $H = -wr^2$ and $D = r$), a_1 and a_2 are not necessarily to be zero. For classes I and III, the Eqs.(5.16) and (5.17) yield

$$g_{,\phi} + \left(\frac{2w}{m} \right)^2 h = \left[1 - \left(\frac{2w}{m} \right)^2 \right] h = 0, \quad (5.18)$$

$$h_{,\phi} - \left(\frac{2w}{m} \right)^2 g = \left[1 - \left(\frac{2w}{m} \right)^2 \right] g = 0. \quad (5.19)$$

Thus, we have two different cases of solutions for classes I and III,

$$(a) \ g = h = 0, \quad \text{where } m^2 \neq 4w^2, \quad (5.20)$$

$$(b) \ g_{,\phi} + h = 0, \quad h_{,\phi} - g = 0, \quad \text{where } m^2 = 4w^2. \quad (5.21)$$

In case (b), the functions g and h have the following solutions

$$g = h_5(\tau) \cos \phi + h_6(\tau) \sin \phi, \quad h = h_5(\tau) \sin \phi - h_6(\tau) \cos \phi, \quad (5.22)$$

where $h_5(\tau)$ and $h_6(\tau)$ are integration functions.

In classes I and III, when $m^2 \neq 4w^2$, i.e. the case (a), the Eqs. (5.14) and (5.15) are satisfied only if

$$L_{1,\phi\phi} + L_1 = DK_{1,rr} - D'K_{1,r} = \text{const.} = q_1, \quad (5.23)$$

$$L_{2,\phi\phi} + L_2 = DK_{2,rr} - D'K_{2,r} = \text{const.} = q_2, \quad (5.24)$$

where q_1, q_2 are separation constants. Integrating (5.23) and (5.24) we obtain

$$L_1 = a_3 \cos \phi + a_4 \sin \phi + q_1, \quad K_1 = -q_1 D + a_5 \int D dr + a_6, \quad (5.25)$$

$$L_2 = c_2 \cos \phi + c_3 \sin \phi + q_2, \quad K_2 = -q_2 D + c_4 \int D dr + c_5, \quad (5.26)$$

where a_i 's and c_i 's are constant parameters. Now the Eq.(4.16) is identically satisfied for classes I and III. The remaining (i.e. second, third and fourth) Eqs. of (4.18) yields $a_3 = a_4 = a_5 = a_6 = 0, h_4 = c_6 \tau + c_7$ and $f = -c_6 z$, where c_6, c_7 are integration constants. It is noted here that there is no contribution of the separation constants q_1 and q_2 to the NGS vector fields. Then the NGS vector field components and the gauge function for class I with the condition $m^2 \neq w^2$ are found as

$$\xi = c_1, \quad \eta^0 = -\frac{H}{D}(c_2 \cos \phi + c_3 \sin \phi) + c_4 \frac{2w}{m} + c_5, \quad \eta^1 = -c_2 \sin \phi + c_3 \cos \phi, \quad (5.27)$$

$$\eta^2 = -\frac{D'}{D}(c_2 \cos \phi + c_3 \sin \phi) - c_4 m, \quad \eta^3 = c_6 \tau + c_7, \quad f = -c_6 z, \quad (5.28)$$

which means that we have *seven* NGSs, i.e. the *five* KV's $\mathbf{X}_1, \dots, \mathbf{X}_5$

$$\begin{aligned} \mathbf{X}_1 &= \partial_t, \quad \mathbf{X}_2 = \partial_z, \quad \mathbf{X}_3 = \frac{2\omega}{m} \partial_t - m \partial_\phi, \\ \mathbf{X}_4 &= -\frac{H}{D} \sin \phi \partial_t + \cos \phi \partial_r - \frac{D'}{D} \sin \phi \partial_\phi, \\ \mathbf{X}_5 &= -\frac{H}{D} \cos \phi \partial_t - \sin \phi \partial_r - \frac{D'}{D} \cos \phi \partial_\phi, \end{aligned} \quad (5.29)$$

and *two non-Killing* NGSs

$$\mathbf{Y}_1 = \partial_\tau, \quad \mathbf{Y}_2 = \tau \partial_z \quad \text{with gauge term } f = -z. \quad (5.30)$$

The Lie algebra has the following non-vanishing commutators:

$$[\mathbf{X}_3, \mathbf{X}_4] = -m \mathbf{X}_5, \quad [\mathbf{X}_4, \mathbf{X}_5] = m \mathbf{X}_3, \quad [\mathbf{X}_5, \mathbf{X}_3] = -m \mathbf{X}_4.$$

It should be noticed that the expressions for all KV's are time-independent.

In class III, where $m^2 \equiv -\mu^2 < 0, \mu^2 > 0$ and $w \neq 0$, it follows that the KV's $\mathbf{X}_1, \mathbf{X}_2, \mathbf{X}_4, \mathbf{X}_5$ and the non-Killing NGSs $\mathbf{Y}_1, \mathbf{Y}_2$ are same form as given the above, but only \mathbf{X}_3 has the form as $(2w/\mu) \partial_t + \mu \partial_\phi$.

For class I the first integrals (4.8) associated with $\mathbf{X}_1, \dots, \mathbf{X}_5, \mathbf{Y}_1$ and \mathbf{Y}_2 are found as

$$I_1 = \dot{t} + H \dot{\phi}, \quad I_2 = -\dot{z}, \quad I_3 = \frac{2w}{m} I_1 - m \left[H \dot{t} + (H^2 - D^2) \dot{\phi} \right], \quad (5.31)$$

$$I_4 = -\frac{\sin \phi}{D} \left\{ H(1 + D') \dot{t} + [H^2 + (H^2 - D^2) D'] \dot{\phi} \right\} - \cos \phi \dot{r}, \quad (5.32)$$

$$I_5 = -\frac{\cos \phi}{D} \left\{ H(1 + D') \dot{t} + [H^2 + (H^2 - D^2) D'] \dot{\phi} \right\} + \sin \phi \dot{r}, \quad (5.33)$$

$$I_6 = -E_L, \quad I_7 = -\tau\dot{z} + z, \quad (5.34)$$

where the E_L is the Hamiltonian (3.9) of the dynamical system and yields

$$E_L = \frac{1}{2} \left\{ I_1^2 - I_2^2 - \frac{1}{D^2} \left[HI_1 - \left(\frac{2w}{m^2} I_1 - \frac{I_3}{m} \right) \right]^2 - \dot{r}^2 \right\}. \quad (5.35)$$

Hence the Hamiltonian E_L is conserved ($\partial_\tau E_L = \frac{dE_L}{d\tau} = 0$). Therefore, the vector field \mathbf{Y}_1 is the trivial NGS. Here the constants of motion $p_t \equiv I_1, p_\phi \equiv 2wI_1/m^2 - I_3/m$ and $p_z \equiv I_2$ represent the conservation of energy, angular momentum and z component of momentum, respectively. It can be solved $\dot{t}, \dot{\phi}, \dot{r}$ and \dot{z} from Eqs. (5.31)-(5.34). Furthermore, we can see from the Eq.(3.2) that $\epsilon = -2I_6$, where $\epsilon = -1, 0, +1$ for spacelike, null and timelike geodesics, respectively. Then the integration for the coordinate z easily give $z = -p_z\tau + I_7$, and we obtain all \dot{x}^a and a constraint equation from the first integrals given above as

$$\dot{t} = \frac{1}{D^2} [p_\phi H + p_t(D^2 - H^2)], \quad (5.36)$$

$$\dot{\phi} = \frac{1}{D^2} [p_t H - p_\phi], \quad (5.37)$$

$$\dot{z} = -p_z, \quad (5.38)$$

$$\dot{r}^2 = p_t^2 - V(r), \quad (5.39)$$

$$\dot{r} = -I_4 \cos \phi + I_5 \sin \phi, \quad (5.40)$$

$$p_t H + p_\phi D' + (I_4 \sin \phi + I_5 \cos \phi) D = 0, \quad (5.41)$$

where we have defined the *effective potential*

$$V(r) := \frac{1}{D^2} (p_t H - p_\phi)^2 + p_z^2 + \epsilon. \quad (5.42)$$

The Eq. (5.39) with the effective potential given in (5.42) is the generalization of radial equation given in Ref. [18]. Furthermore, we have seen that there exist a new radial equation, the Eq. (5.40), which depends only on ϕ , not r . Differentiating (5.40) with respect to τ and using $\dot{\phi}$ of (5.37) and the constraint Eq. (5.41) we find the following Lienard type differential equation [49]

$$\ddot{r} = -\frac{1}{D^3} (p_t H - p_\phi) (p_t H + p_\phi D'). \quad (5.43)$$

The substitution of $W(r) = \dot{r}$ leads to an Abel differential equation of the second kind as

$$WW' = -\frac{1}{D^3} (p_t H - p_\phi) (p_t H + p_\phi D'), \quad (5.44)$$

which can be written as

$$(W^2)' = -\left[\frac{1}{D^2} (p_t H - p_\phi)^2 \right]', \quad (5.45)$$

yielding

$$W^2 \equiv \dot{r}^2 = -\frac{1}{D^2} (p_t H - p_\phi)^2 + r_0, \quad (5.46)$$

which is explicitly equivalent to the radial equation (5.39), where $r_0 = p_t^2 - p_z^2 - \epsilon$. Introducing a new variable $u = m^2 H/4w$ which is equivalent to $\sinh^2(mr/2)$ for the class I (see the Eq. (42) of ref. [18]), the Eq. (5.39) gives

$$\dot{u}^2 = m^2 p_t^2 \left[-\eta u^2 + (1 - \beta^2 + 2w\gamma)u - \frac{m^2}{4} \gamma^2 \right], \quad (5.47)$$

where η , β^2 and γ are defined as

$$\eta = \beta^2 - 1 + \frac{4w^2}{m^2}, \quad \beta^2 = \frac{p_z^2 + \epsilon}{p_t^2}, \quad \gamma = \frac{p_\phi}{p_t}. \quad (5.48)$$

Using the above definitions the effective potential takes the form of

$$V(r) = \frac{p_t^2}{D^2} (H - \gamma^2)^2 + \beta^2 p_t^2. \quad (5.49)$$

Therefore from the radial equation (5.39), one can accomplish a complete characterization of the motion which depends essentially on the parameters β, γ, m and w . This characterization separates the motion into three distinct cases $\gamma > 0$, $\gamma = 0$, and $\gamma < 0$. For the trajectories of physical particles, it follows from the Eq. (5.39) that $0 \leq \beta^2 \leq 1$ (see also Refs. [14, 18]).

The general solution of (5.47) is given by

$$u(\tau) = \frac{1}{2\eta} \left[1 - \beta^2 + 2w\gamma + \sqrt{(1 - \beta^2 + 2w\gamma)^2 - \eta m^2 \gamma^2} \sin(mp_t \sqrt{\eta}(\tau - \tau_0)) \right], \quad (5.50)$$

where $\eta \neq 0$ and $(1 - \beta^2 + 2w\gamma)^2 - \eta m^2 \gamma^2 \geq 0$. Using the new variable $u = m^2 H / 4w$ in Eqs. (5.36) and (5.37) for $t(\tau)$ and $\phi(\tau)$ we find

$$\dot{t} = \frac{p_t(1 + w\gamma)}{u} + p_t \left(1 - \frac{4w^2}{m^2} \right) \frac{u}{1 + u}, \quad (5.51)$$

$$\dot{\phi} = \frac{wp_t}{1 + u} - \frac{p_t m^2 \gamma / 4}{u(1 + u)}, \quad (5.52)$$

Now we shall use the solution (5.50) of the radial equation to solve the above equations for $t(\tau)$ and $\phi(\tau)$. For $\gamma \neq 0$, after some algebra, we have obtained the following general solutions of the Eqs.(5.51) and (5.52)

$$t(\tau) = \frac{2w(\gamma + 4w/m^2)}{m\sqrt{\eta}\sqrt{(1+p)^2 - q^2}} \arctan \left[\frac{(1+p) \tan(mp_t \sqrt{\eta}(\tau - \tau_0)/2) + q}{\sqrt{(1+p)^2 - q^2}} \right] + p_t \left(1 - \frac{4w^2}{m^2} \right) \tau + t_0, \quad (5.53)$$

$$\phi(\tau) = \frac{m(\gamma + 4w/m^2)}{2\sqrt{\eta}\sqrt{(1+p)^2 - q^2}} \arctan \left[\frac{(1+p) \tan(mp_t \sqrt{\eta}(\tau - \tau_0)/2) + q}{\sqrt{(1+p)^2 - q^2}} \right] - \arctan \left[\frac{2\sqrt{\eta}}{m\gamma} \{p \tan(mp_t \sqrt{\eta}(\tau - \tau_0)/2) + q\} \right] + \phi_0, \quad (5.54)$$

where $(1+p)^2 > q^2$, $t_0 = t(0)$ and $\phi_0 = \phi(0)$ are integration constants. Here we have introduced the parameters p and q as

$$p := \frac{1 - \beta^2 + 2w\gamma}{2\eta}, \quad q := \sqrt{p^2 - \frac{m^2 \gamma^2}{4\eta}}, \quad (5.55)$$

where $p^2 \geq m^2 \gamma^2 / 4\eta$. For $\gamma = 0$, the general solution of the Eq. (5.47) yields

$$u(\tau) = \frac{(\beta^2 - 1)}{2\eta} [-1 + \sin(mp_t \sqrt{\eta}(\tau - \tau_0))], \quad (5.56)$$

and using this solution in Eqs. (5.51) and (5.52) it follows that

$$t(\tau) = \frac{8w^2}{m^3\sqrt{\eta(1+2p)}} \arctan \left[\frac{(1+p)\tan(mp_t\sqrt{\eta}(\tau-\tau_0)/2) - p}{\sqrt{1+2p}} \right] + p_t \left(1 - \frac{4w^2}{m^2} \right) \tau + t_0, \quad (5.57)$$

$$\phi(\tau) = \frac{2w}{m\sqrt{\eta(1+p)}} \arctan \left[\frac{(1+p)\tan(mp_t\sqrt{\eta}(\tau-\tau_0)/2) - p}{\sqrt{1+2p}} \right] + \phi_0, \quad (5.58)$$

where $p = (1 - \beta^2)/2\eta$.

In the special class I case, where $m^2 = 4\omega^2$, i.e. $m = +2w$, which comes from the case (b), Eq.(4.16) is also identically satisfied, and the remaining (i.e. second, third and fourth) Eqs. of (4.18) give $a_3 = a_4 = a_5 = a_6 = 0$, $h_4(\tau) = c_6\tau + c_7$, $h_5 = c_8$, $h_6 = c_9$, and $f = -c_6z$, where c_6, c_7, c_8, c_9 are integration constants. Hence the quantities P, R, T, Z have the form

$$P = c_8 \cos(2wt + \phi) + c_9 \sin(2wt + \phi) - D'(c_2 \cos \phi + c_3 \sin \phi) - c_4 \frac{D}{2w}, \quad (5.59)$$

$$R = c_8 \sin(2wt + \phi) - c_9 \cos(2wt + \phi) - c_2 \sin \phi + c_3 \cos \phi, \quad (5.60)$$

$$T = 2wD(c_2 \cos \phi + c_3 \sin \phi) + c_4 \int Ddr + c_5, \quad (5.61)$$

$$Z = c_6\tau + c_7. \quad (5.62)$$

Thus using the definition (4.19) we find

$$\begin{aligned} \xi &= c_1, \quad \eta^0 = -\frac{H}{D}(c_2 \cos \phi + c_3 \sin \phi) + c_4 \frac{2w}{m} - \frac{H}{D}[c_8 \cos(mt + \phi) + c_9 \sin(mt + \phi)] + c_5, \\ \eta^1 &= -c_2 \sin \phi + c_3 \cos \phi + c_8 \sin(mt + \phi) - c_9 \cos(mt + \phi), \\ \eta^2 &= -\frac{D'}{D}(c_2 \cos \phi + c_3 \sin \phi) + \frac{1}{D}[c_8 \cos(mt + \phi) + c_9 \sin(mt + \phi)] - c_4m, \\ \eta^3 &= c_6\tau + c_7, \quad f = -c_6z. \end{aligned} \quad (5.63)$$

Then one finds that there are *nine* NGSs which are *seven* KVs $\mathbf{X}_1, \dots, \mathbf{X}_7$ given by

$$\begin{aligned} \mathbf{X}_1 &= \partial_t, \quad \mathbf{X}_2 = \partial_z, \quad \mathbf{X}_3 = \partial_t - m\partial_\phi, \\ \mathbf{X}_4 &= -\frac{H}{D}\sin\phi\partial_t + \cos\phi\partial_r - \frac{D'}{D}\sin\phi\partial_\phi, \\ \mathbf{X}_5 &= -\frac{H}{D}\cos\phi\partial_t - \sin\phi\partial_r - \frac{D'}{D}\cos\phi\partial_\phi, \\ \mathbf{X}_6 &= -\frac{H}{D}\cos(mt + \phi)\partial_t + \sin(mt + \phi)\partial_r + \frac{1}{D}\cos(mt + \phi)\partial_\phi, \\ \mathbf{X}_7 &= -\frac{H}{D}\sin(mt + \phi)\partial_t - \cos(mt + \phi)\partial_r + \frac{1}{D}\sin(mt + \phi)\partial_\phi, \end{aligned} \quad (5.64)$$

and *two non-Killing* NGSs given by (5.30), where $m = +2\omega$. The corresponding Lie algebra has the following non-vanishing commutators:

$$\begin{aligned} [\mathbf{X}_3, \mathbf{X}_4] &= -m\mathbf{X}_5, \quad [\mathbf{X}_4, \mathbf{X}_5] = m\mathbf{X}_3, \quad [\mathbf{X}_5, \mathbf{X}_3] = -m\mathbf{X}_4, \\ [\mathbf{X}_1, \mathbf{X}_6] &= -m\mathbf{X}_7, \quad [\mathbf{X}_6, \mathbf{X}_7] = m\mathbf{X}_1, \quad [\mathbf{X}_1, \mathbf{X}_7] = m\mathbf{X}_6. \end{aligned}$$

The first integrals (5.31)-(5.34) for $\mathbf{X}_1, \dots, \mathbf{X}_5, \mathbf{Y}_1$ and \mathbf{Y}_2 together with $m = +2w$ are same ones for this special class I. The remaining first integrals associated with \mathbf{X}_6 and \mathbf{X}_7 given in (5.64) are

$$I_8 = -\frac{\cos(mt + \phi)}{D} \left\{ \left(H - \frac{2w}{m^2} \right) I_1 + \frac{I_3}{m} \right\} - \sin(mt + \phi) \dot{r}, \quad (5.65)$$

$$I_9 = -\frac{\sin(mt + \phi)}{D} \left\{ \left(H - \frac{2w}{m^2} \right) I_1 + \frac{I_3}{m} \right\} + \cos(mt + \phi) \dot{r}. \quad (5.66)$$

which yields

$$\dot{r} = -I_8 \sin(mt + \phi) + I_9 \cos(mt + \phi), \quad (5.67)$$

$$p_t H - p_\phi + [I_8 \cos(mt + \phi) + I_9 \sin(mt + \phi)] D = 0. \quad (5.68)$$

For the class II, where $H(r) = -wr^2$ and $D(r) = r$, as earlier mentioned from the Eqs. (5.14) and (5.15), the constant parameters a_1, a_2 are not necessarily to be zero. But in this class, the Eq. (4.16) is not identically satisfied, and gives $a_1 = a_2 = 0$ and it follows from the Eq. (4.15) that the functions g and h are also vanishing. The remaining part of calculation of the NGS vector field components is similar to the case of class I. After rearranging the constant parameters, the NGS vector field components and the gauge function of class II are obtained as

$$\xi = c_1, \quad \eta^0 = wr(c_2 \cos \phi + c_3 \sin \phi) + c_4, \quad \eta^1 = -c_2 \sin \phi + c_3 \cos \phi, \quad (5.69)$$

$$\eta^2 = -\frac{1}{r}(c_2 \cos \phi + c_3 \sin \phi) + c_5, \quad \eta^3 = c_6 \tau + c_7, \quad f = -c_6 \tau, \quad (5.70)$$

which give *seven* NGSs, i.e. the *five* KVs $\mathbf{X}_1, \dots, \mathbf{X}_5$ by

$$\begin{aligned} \mathbf{X}_1 &= \partial_t, \quad \mathbf{X}_2 = \partial_z, \quad \mathbf{X}_3 = \partial_\phi, \\ \mathbf{X}_4 &= -\omega r \sin \phi \partial_t - \cos \phi \partial_r + \frac{1}{r} \sin \phi \partial_\phi, \\ \mathbf{X}_5 &= -\omega r \cos \phi \partial_t + \sin \phi \partial_r + \frac{1}{r} \cos \phi \partial_\phi \end{aligned} \quad (5.71)$$

and *two non-Killing* NGSs same as (5.30). The Lie algebra will have the following non-vanishing commutators:

$$[\mathbf{X}_3, \mathbf{X}_4] = \mathbf{X}_5, \quad [\mathbf{X}_3, \mathbf{X}_5] = -\mathbf{X}_4, \quad [\mathbf{X}_4, \mathbf{X}_5] = 2\omega \mathbf{X}_1.$$

Hence, the first integrals associated with those vector fields are

$$\begin{aligned} I_1 &= \dot{t} - wr^2 \dot{\phi}, \quad I_2 = -\dot{z}, \quad I_3 = -wr^2 \dot{t} + r^2(w^2 r^2 - 1) \dot{\phi}, \\ I_4 &= \sin \phi \left[wr I_1 - \frac{1}{r} I_3 \right] - \dot{r} \cos \phi, \\ I_5 &= \cos \phi \left[wr I_1 - \frac{1}{r} I_3 \right] + \dot{r} \sin \phi, \\ I_6 &= \frac{1}{2} \left[-\dot{t}^2 + \dot{r}^2 + r^2(1 - w^2 r^2) \dot{\phi}^2 + \dot{z}^2 \right] + wr^2 \dot{t} \dot{\phi}, \\ I_7 &= -\tau \dot{z} + z, \end{aligned} \quad (5.72)$$

where $I_6 = -E_L$ and $\epsilon = -2I_6$ which takes the values $-1, 0$ and 1 for spacelike, null and timelike geodesics, respectively. Using the obtained first integrals given above, it follows that the coordinate z is $z(\tau) = -p_z\tau + I_7$ and

$$\dot{t} = p_t (1 - w^2 r^2) - p_\phi w, \quad (5.73)$$

$$\dot{\phi} = -wp_t - \frac{p_\phi}{r^2}, \quad (5.74)$$

$$\dot{r}^2 = p_t^2 - V(r), \quad (5.75)$$

$$\dot{r} = -I_4 \cos \phi + I_5 \sin \phi, \quad (5.76)$$

$$I_4 \sin \phi + I_5 \cos \phi = wp_t r - \frac{p_\phi}{r}, \quad (5.77)$$

where $p_t = I_1$, $p_z = I_2$ and $p_\phi = I_3$ are constants of motion related with energy, z component of momentum and angular momentum, respectively, and the effective potential is defined as

$$V(r) = \left[wp_t r + \frac{p_\phi}{r} \right]^2 + p_z^2 + \epsilon. \quad (5.78)$$

It is easily seen that one can derive the radial equation (5.75) from Eqs. (5.74), (5.76) and (5.77). Introducing the new variable $v = r^2$ the radial Eq. (5.75) becomes

$$\dot{v}^2 = -4w^2 p_t^2 v^2 + 4\alpha v - 4p_\phi^2, \quad (5.79)$$

with α defined by $\alpha = p_t^2 - 2wp_t p_\phi - p_z^2 - \epsilon$. The general solution of (5.79) is

$$v(\tau) \equiv r^2 = \frac{\alpha}{2w^2 p_t^2} + \sqrt{\alpha^2 - 4w^2 p_t^2 p_\phi^2} \sin[2wp_t(\tau - \tau_0)], \quad (5.80)$$

where $\alpha \geq \pm 2wp_t p_\phi$. Then after substitution (5.80) into (5.73) the general solution of the resulting differential equation for $t(\tau)$ gives

$$t(\tau) = \left(p_t - wp_\phi - \frac{\alpha}{2p_t} \right) \tau + \frac{w}{2} \sqrt{\alpha^2 - 4w^2 p_t^2 p_\phi^2} \cos[2wp_t(\tau - \tau_0)] + t_0. \quad (5.81)$$

Finally, using (5.80) in (5.74) the behavior of the coordinate ϕ is given by

$$\phi(\tau) = -wp_t \tau + \frac{2wp_t p_\phi}{\sqrt{4\beta^2 w^4 p_t^4 - \alpha^2}} \tanh^{-1} \left\{ \frac{\alpha \tan[wp_t(\tau - \tau_0)] + 2\beta w^2 p_t^2}{\sqrt{4\beta^2 w^4 p_t^4 - \alpha^2}} \right\} + \phi_0, \quad (5.82)$$

where $\beta = \sqrt{\alpha^2 - 4w^2 p_t^2 p_\phi^2}$ and $4\beta^2 w^4 p_t^4 > \alpha^2 \geq 4w^2 p_t^2 p_\phi^2$. For a special case such as $p_\phi = I_3 = 0$ the first integrals simplify considerably. In the latter special case the geodesic equations can be integrated completely and the solution reads

$$t(\tau) = \left[p_t - \frac{1}{2p_t} (I_4^2 + I_5^2) \right] \tau + \frac{1}{4wp_t^2} [(I_4^2 - I_5^2) \sin(2wp_t \tau) - 4I_4 I_5 \cos^2(wp_t \tau)] + t_0, \quad (5.83)$$

$$r(\tau) = \frac{1}{wp_t} [I_5 \cos(wp_t \tau) - I_4 \sin(wp_t \tau)], \quad (5.84)$$

$$\phi(\tau) = -wp_t \tau + \phi_0, \quad (5.85)$$

$$z(\tau) = -p_z \tau + I_7, \quad (5.86)$$

where $p_t^2 - (I_2^2 + I_4^2 + I_5^2) = \epsilon$, and ϕ_0 is an integration constant.

5.2 Class IV

In this class, where $m^2 \neq 0, w = 0$, the metric functions are taken as $H(r) = 0$ and $D(r) = \frac{1}{m} \sinh(mr)$ for $m^2 > 0$, or $D(r) = \frac{1}{\mu} \sin(\mu r)$ for $\mu^2 = -m^2 > 0$. Here the functions T, R, Z given by the Eqs. (5.1)-(5.3), which are the general solution to the NGS equations (4.9)-(4.11), have the same form. For this class the equation (4.12) yields that $h_1 = 0$ and

$$f_1 = rh_5(\tau, \phi) + h_6(\tau, \phi), \quad g_1 = th_5(\tau, \phi) + h_7(\tau, \phi), \quad (5.87)$$

where h_5, h_6, h_7 are integration functions of their arguments. For vanishing potential, the Eq. (4.17) and first equation of (4.18) give

$$\xi = c_1 + b_1\tau + b_2\frac{\tau^2}{2}, \quad (5.88)$$

$$f = b_2\frac{t^2}{4} + t[zL_1(\phi) + rL_3(\phi) + L_5(\phi)] + f_2(r, \phi, z), \quad (5.89)$$

$$h_2 = \tau L_1(\phi) + L_2(\phi), \quad h_5 = \tau L_3(\phi) + L_4(\phi), \quad h_6 = \tau L_5(\phi) + L_6(\phi) \quad (5.90)$$

where $L_i(\phi)$'s and $f_2(r, \phi, z)$ are integration functions, b_1 and b_2 are integration constants. Thus the function P follows from the Eq. (4.13) as

$$P = \frac{z}{D} [-t(\tau L_{1,\phi} + L_{2,\phi}) + rh_{3,\phi} - h_{4,\phi}] + K(\tau, t, r, \phi), \quad (5.91)$$

where $K(\tau, t, r, \phi)$ is an integration function. Using the functions P and T in Eq. (4.14) reads $L_1 = a_1, L_2 = a_2$ and

$$K = \frac{t}{D} [rh_{5,\phi} + h_{6,\phi}] + f_3(\tau, r, \phi), \quad (5.92)$$

where f_3 is a function of integration, and a_1, a_2 are constants. Using the Eq. (4.15) one gets that h_3 and h_4 are only depend on τ , and $L_3 = a_3, L_4 = a_4, L_5 = a_5, L_6 = a_6$, where a_1, \dots, a_6 are constants, and $f_3 = D'h_{7,\phi} + Dh_8(\tau, \phi)$, h_8 is an integration function. Inserting the obtained results into Eq. (4.16), after some algebra, we get that $a_3 = a_4 = b_1 = b_2 = 0, h_3 = 0, h_7 = k_1(\tau) \cos \phi + k_2(\tau) \sin \phi$ and $h_8 = k_3(\tau)$, in which the quantities k_1, k_2 and k_3 are integration functions. Finally, the remaining three equations of (4.18) yield $a_1 = 0, k_1 = a_7, k_2 = a_8, k_3 = a_9, h_4 = a_{10}\tau + a_{11}$ and $f_2 = -a_{10}\tau + a_{12}$, where a_7, \dots, a_{12} are constants. Putting these results into the functions P, R, T, Z and f , they take the following form

$$P = D'(-a_7 \sin \phi + a_8 \cos \phi) + a_9 D, \quad (5.93)$$

$$R = a_7 \cos \phi + a_8 \sin \phi, \quad (5.94)$$

$$T = a_2 z + a_5 \tau + a_6, \quad (5.95)$$

$$Z = a_2 t a_{10} \tau + a_{11}, \quad f = a_5 t - a_{10} z. \quad (5.96)$$

Rearranging the constant parameters, the NGS vector fields and the gauge function for this class are

$$\xi = c_1, \quad \eta^0 = c_2 z + c_3 s + c_4, \quad \eta^1 = c_5 \cos \phi + c_6 \sin \phi, \quad (5.97)$$

$$\eta^2 = \frac{D'}{D}(-c_5 \sin \phi + c_6 \cos \phi) + c_7, \quad \eta^3 = c_2 t + c_8 \tau + c_9, \quad f = c_3 t - c_8 \tau. \quad (5.98)$$

This result yields *nine* NGSs, i.e., the *six* KVs $\mathbf{X}_1, \dots, \mathbf{X}_6$ by

$$\begin{aligned}\mathbf{X}_1 &= \partial_t, & \mathbf{X}_2 &= \partial_z, & \mathbf{X}_3 &= z \partial_t + t \partial_z, \\ \mathbf{X}_4 &= \cos \phi \partial_r - \frac{D'}{D} \sin \phi \partial_\phi, \\ \mathbf{X}_5 &= -\sin \phi \partial_r - \frac{D'}{D} \cos \phi \partial_\phi, & \mathbf{X}_6 &= \partial_\phi,\end{aligned}\tag{5.99}$$

two non-Killing NGSs same as (5.30), and extra one non-Killing NGS

$$\mathbf{Y}_3 = \tau \partial_t \quad \text{with gauge term } f = t.\tag{5.100}$$

The non-vanishing commutators of NGSs are

$$\begin{aligned}[\mathbf{X}_1, \mathbf{X}_3] &= \mathbf{X}_2, & [\mathbf{X}_2, \mathbf{X}_3] &= \mathbf{X}_1, \\ [\mathbf{X}_4, \mathbf{X}_5] &= -m^2 \mathbf{K}_6, & [\mathbf{X}_5, \mathbf{X}_6] &= \mathbf{X}_4, & [\mathbf{X}_6, \mathbf{X}_4] &= \mathbf{X}_5.\end{aligned}$$

If $m^2 = \omega = 0$, then the line element (2.1) is clearly Minkowskian. Therefore, this particular case has not been included in this study. The first integrals of this class are

$$\begin{aligned}I_1 &= \dot{t}, & I_2 &= -\dot{z}, & I_3 &= z\dot{t} - t\dot{z}, \\ I_4 &= -\dot{r} \cos \phi + \dot{\phi} D D' \sin \phi, \\ I_5 &= \dot{r} \sin \phi + \dot{\phi} D D' \cos \phi, \\ I_6 &= -D^2 \dot{\phi}, & I_7 &= \frac{1}{2} \left[-\dot{t}^2 + \dot{r}^2 + D^2 \dot{\phi}^2 + \dot{z}^2 \right], \\ I_8 &= -\tau \dot{z} + z, & I_9 &= \tau \dot{t} - t,\end{aligned}\tag{5.101}$$

where $I_7 = -E_L$. From the Eq.(3.2), we get that $\epsilon = -2I_7$, and $\epsilon = -1, 0, 1$ for spacelike, lightlike and timelike geodesics, respectively. The above constants of motion can be solved for $\dot{t}, \dot{r}, \dot{\phi}$, and \dot{z} . Then, we find from (5.101) that $t = \tau I_1 - I_9$ and $z = -I_2 \tau + I_8$, which means that along any geodesic the time coordinate t and the axial coordinate z vary uniformly with respect to its affine parameter τ . It also follows from (5.101) that

$$\dot{\phi} = -\frac{I_6}{D^2},\tag{5.102}$$

$$\dot{r}^2 = I_1^2 - I_2^2 + 2I_7 - \frac{I_6^2}{D^2},\tag{5.103}$$

$$\dot{r} = -I_4 \cos \phi + I_5 \sin \phi,\tag{5.104}$$

$$I_4 \sin \phi + I_5 \cos \phi + \frac{D'}{D} I_6 = 0,\tag{5.105}$$

$$I_1 I_8 - I_2 I_9 = 0.\tag{5.106}$$

Here, the constants of motion representing conservation of energy, angular momentum and z component of the momentum are $p_t = I_1, p_\phi = I_6$ and $p_z = I_2$, respectively. Thus, we can write the Eq. (5.103) as

$$\dot{r}^2 = p_t^2 - V(r),\tag{5.107}$$

where the effective potential is defined by

$$V(r) = \frac{p_\phi^2}{D^2} + p_z^2 + \epsilon.\tag{5.108}$$

Differentiating \dot{r} given in the Eq. (5.104) with respect to proper time τ and using (5.102) and (5.105), we obtain

$$\ddot{r} = p_\phi^2 \frac{D'}{D^3}, \quad (5.109)$$

which is again a Lienard type differential equation. By taking $\dot{D} = D'\dot{r}$, the integration of (5.109) with respect to τ is reduced to the same form with (5.103). Now, we introduce a new variable $\sigma = m^2 D^2 = \sinh^2(mr)$ for $m^2 > 0$. Thus the Eq. (5.107) in terms of the new variable yields

$$\dot{\sigma}^2 = 4m^2 p_t^2 [-\eta\sigma^2 - (\eta + m^2\gamma^2)\sigma - m^2\gamma^2], \quad (5.110)$$

where $\eta = (p_z^2 + \epsilon)/p_t^2 - 1$ and $\gamma = p_\phi/p_t$. The general solution of this equation is

$$\sigma(\tau) = \frac{1}{2\eta} [-(\eta + m^2\gamma^2) + (\eta - m^2\gamma^2) \sin(2mp_t\sqrt{\eta}(\tau - \tau_0))]. \quad (5.111)$$

Considering this new variable in the Eq. (5.102) for $\phi(\tau)$ we get

$$\dot{\phi} = -\frac{m^2 p_\phi}{\sigma}, \quad (5.112)$$

which has the general solution as

$$\phi(\tau) = -\arctan \left[\frac{\eta - m^2\gamma^2 - (\eta + m^2\gamma^2) \tan(mp_t\sqrt{\eta}(\tau - \tau_0))}{2m\gamma\sqrt{\eta}} \right] + \phi_0. \quad (5.113)$$

When the new variable σ has the form $\sigma = \mu^2 D^2 = \sin^2(\mu r)$ for $\mu^2 = -m^2 > 0$, the Eq. (5.107) give

$$\dot{\sigma}^2 = 4\mu^2 p_t^2 [\eta\sigma^2 - (\eta - \mu^2\gamma^2)\sigma - \mu^2\gamma^2], \quad (5.114)$$

and the general solution of this equation is

$$\sigma(\tau) = \frac{1}{8\eta^{3/2}} \left[4\sqrt{\eta}(\eta - \mu^2\gamma^2) + (\mu^2\gamma^2 - \eta)e^{2\mu p_t\sqrt{\eta}(\tau - \tau_0)} + 4\eta e^{-2\mu p_t\sqrt{\eta}(\tau - \tau_0)} \right]. \quad (5.115)$$

Then the Eq. (5.102) for $\phi(\tau)$ has the general solution

$$\phi(\tau) = \frac{-2\mu\gamma\sqrt{\eta}}{\sqrt{(\mu^2\gamma^2 - \eta)(1 + \mu^2\gamma^2 - \eta)}} \arctan \left[\frac{1}{2} \sqrt{\frac{(\mu^2\gamma^2 - \eta)}{\eta(1 + \mu^2\gamma^2 - \eta)}} \left(e^{2\mu p_t\sqrt{\eta}(\tau - \tau_0)} - 2\sqrt{\eta} \right) \right] + \phi_0, \quad (5.116)$$

where $\mu^2\gamma^2 > 0$ and $\eta(1 + \mu^2\gamma^2 - \eta) > 0$.

6 Conclusions

In this paper, we have obtained the NGSs of geodesic Lagrangian L for Gödel-type spacetimes for classes I, II, III and IV for which we have found 7 NGS generators. Thus, the Gödel-type spacetimes corresponding to those classes admit the algebra $\mathcal{N}_7 \supset \mathcal{G}_5$. In special class I (where $m^2 = 4w^2$) and class IV, we have found 9 NGS generators. The NGS algebra admitted by the special class I is $\mathcal{N}_9 \supset \mathcal{G}_7$. The Gödel-type spacetime in class IV admits the algebra $\mathcal{N}_9 \supset \mathcal{G}_6$.

We obtained the first integrals admitted by geodesic Lagrangians for the Gödel-type spacetimes of each class I, II, III and IV, that are due to the existence of NGS vector fields

including the KVs. Using the obtained first integrals in all classes of Gödel-type spacetimes, we have derived the analytical solutions of geodesic equations which represents the usefulness of the NGSs. As stated in the literature [18, 19] that the radial equation of the geodesic motion is depend on the radial coordinate and its derivative, for example the Eq. (5.39) for class I. But, in each classes of Gödel-type spacetimes using the NGSs found in this study, we derive a radial equation which depends only on angular coordinate ϕ (the Eq. (5.40) for class I; the Eq. (5.76) for class II, and the Eq. (5.104) for class IV), and also a constraint equation (the Eq. (5.41) for class I; the Eq. (5.77) for class II, the Eq. (5.105) for class IV). This is a new and unknown property of geodesic motions for Gödel-type spacetimes. The behaviour of the geodesics in Gödel's universe has been extensively examined by several authors [14, 18, 19], therefore, this behaviour is not a subject of this study.

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